

The Gaussian Integral



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$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

The **Gaussian** integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$ is a very known integral that famously occurs in applied mathematics and exactly in the **Gaussian** (also called normal) continuous probability distribution

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where σ is the standard deviation and μ is the mean of the distribution . The expression is an example of a probability density function(pdf). In probability theory, the expression $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is known also under the name Gauss–Laplace distribution and we know that

$$\Phi(x) = \int_{-\infty}^x f(t)dt$$

So the probability P of an event in the range $X \leq x$ is given by

$$P(X \leq x) = \Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

When $\mu = 0$ and $\sigma = 1$ we the standard normal distribution and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$$

Gauss mentioned the integral in 1809 and said that it was due to French mathematician Pierre-Simon **Laplace**(1749 – 1827), in fact **Laplace** introduced it in 1780 and it appears that it was already known before him. Then, **Laplace** reconsidered it again in 1810 to model natural phenomena. The name normal is due to English Karl **Pearson**(1857 – 1936) in 1893. As it is known in probability theory, the sum $\sum p_i$ is 1, hence

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dx = 1$$

French mathematician Siméon **Poisson**(1781 – 1840) one of **Laplace** student, did work on such improper integrals, so the integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx$$

in some references is referred to as the Poisson integral. The similar integral $\int e^{-x^2} dx$ appears for example in statistics and probability, signal processing, quantum mechanics, electromagnetism, partial differential equations,...

○ The Error function

The integral $\int e^{-x^2} dx$ is not elementary (with infinite number of terms). More other examples known in Analysis are

$$\int e^{x^2} dx, \quad \int \sqrt{1 - \alpha^2 \sin^2(x)} dx \quad (\alpha < 1), \quad \int \frac{\cos(x)}{x} dx, \quad \int \frac{\sin(x)}{x} dx$$

$$\int x^x dx, \quad \int \frac{dx}{\log(x)}, \quad \int e^{-1/x^2} dx \quad \dots$$

So instead of integrating directly we use power series

$$\int e^{-x^2} dx = \int \left(\sum_0^{+\infty} \frac{(-x^2)^n}{n!} \right) dx = \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots \right) dx$$

We can now integrate term by term

$$\int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots \right) dx = x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots$$

$$= \sum_{n=0} \frac{x^{2n+1}}{n! (2n+1)} (-1)^n$$

The above infinite series does not correspond to any known function; it looks to be between two known power series, namely

$$\sin(x) = \sum_0^{+\infty} \frac{x^{2n+1}}{(2n+1)!} (-1)^n$$

and

$$\arctan(x) = \sum_0^{+\infty} \frac{x^{2n+1}}{2n+1} (-1)^n$$

We note that e^{-x^2} can be written e^{iix^2} which gives

$$\cosh(x^2) - \sinh(x^2) = \sum_{n=0}^{+\infty} \frac{x^{4n}}{(2n)!} - \sum_{n=0}^{+\infty} \frac{x^{4n+2}}{(2n+1)!}$$

the two series converge so

$$e^{-x^2} = \sum_{n=0}^{+\infty} \left(\frac{x^{4n}}{(2n)!} - \frac{x^{4n+2}}{(2n+1)!} \right)$$

then, we have then necessarily

$$\sum_{n=0}^{+\infty} \left(\frac{x^{4n}}{(2n)!} - \frac{x^{4n+2}}{(2n+1)!} \right) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{n! (2n+1)} (-1)^n$$

For alternating series, we know that the error of the sum is less than the first omitted term. Let us add the number $\frac{2}{\sqrt{\pi}}$ to the series and name it

$$\frac{2}{\sqrt{\pi}} \sum_{n=0} \frac{x^{2n+1}}{n! (2n+1)} (-1)^n = \text{Error}(x)$$

Some prefer to define it otherwise but equivalently

$$Error[t] = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$$

A relation with Φ is

$$Error[t] = 2\Phi(\sqrt{2}t) - 1$$

Convergence

Before we expose the methods to calculate the improper integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$, we have first to examine if the integral really converges. This form of improper integrals occur often in mathematical Analysis like the domain of **Fourier** analysis, and the problem with this type of integrals is that $\int_{-\infty}^{+\infty} f(x) dx$ may converge in a weird sense such that

$$\int_0^{+\infty} f(x) dx + \int_{-\infty}^0 f(x) dx \neq \int_{-\infty}^{+\infty} f(x) dx$$

so in this case we consider the converges in the sense of the Cauchy principal value noted usually by the symbol PV and means

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{m \rightarrow +\infty} \int_{-m}^m f(x) dx$$

However, in our case we see that the function e^{-x^2} is without singularities in the domain(interval) of integration $] -\infty, +\infty[$, it is defined, continuous and bounded on $] -\infty, +\infty[$. Furthermore, we notice that $f(-x) = f(x)$, so the function under integration is even and

$$\int_{-\infty}^0 e^{-x^2} dx = - \int_0^{-\infty} e^{-x^2} dx = - \int_0^{+\infty} e^{-(-x)^2} d(-x) = \int_0^{+\infty} e^{-x^2} dx$$

so

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = 2 \int_0^{+\infty} e^{-x^2} dx$$

The integral $\int_0^1 e^{-x^2} dx$ is finite, and on $[1, +\infty[$ we have

$$\int_1^{+\infty} |e^{-x^2}| dx = \int_1^{+\infty} e^{-x^2} dx \leq \int_1^{+\infty} e^{-x} dx = 1$$

so the integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$ converges. It is impossible to find the value of $\int_{-\infty}^{+\infty} e^{-x^2} dx$ by using the anti-derivative of e^{-x^2} because $\int e^{-x^2} dx$ is not elementary as we have seen.

So here, I have chosen five methods among those used by mathematicians to evaluate the above integral. We will find out that two methods depend on the value of the integral

$$\int_0^{+\infty} \frac{dx}{1+x^2}$$

The reader can by himself notice that there is a similarity between of the graphs of the two functions e^{-x^2} and $\frac{1}{1+x^2}$.

The double integral

We first emit a result in the theory of double integrals on a domain $D = D_1 \times D_2$:

If $h(x, y) = f(x)g(y)$ then

$$\int_{D_2} \int_{D_1} h(x, y) dx dy = \int_{D_2} \int_{D_1} f(x)g(y) dx dy = \int_{D_2} f(x) dx \int_{D_1} g(y) dy$$

So $\int_{D_2} f(x) dx$ is just a constant inside the second $\int_{D_1} g(y) dy$ and we have

$$\int_{D_2} \int_{D_1} h(x, y) dx dy = \int_{D_1} \int_{D_2} h(x, y) dy dx$$

or

$$\int_{D_1} \left(\int_{D_2} h(x, y) dx \right) dy = \int_{D_2} \left(\int_{D_1} h(x, y) dy \right) dx$$

This is called the **Fubini** theorem, proved by Italian mathematician Guido **Fubini** (1879 – 1943) in 1907, it confirms that we can change (invert) the role of x and y in the double integral without altering the value of the convergent integral¹.

The variables are just dumb symbols we have

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-y^2} dy$$

Putting

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx$$

we have

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$$

¹ This inversion have been known long before **Fubini**, but in the case of continuous function $h(x, y)$. **Fubini** has proved it when $h(x, y)$ is not only continuous but measurable, in conformity with the new integration theory of French mathematician Henri **Lebesgue** (1875 – 1941). The same inversion result for the **Riemann** integral was first established by Austrian mathematician Otto **Stolz** (1842 – 1905) in 1886.

and we have by the same reasoning

$$I^2 = 4 \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dx dy$$

So we have shifted from the calculation of a surface $\int_{-\infty}^{+\infty} e^{-x^2} dx$ to the calculation of a volume $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$.

 The Jacobian

Let

$$\begin{cases} y_1 = f_1(x_1, \dots, x_k) \\ \vdots \\ y_n = f_n(x_1, \dots, x_k) \end{cases}$$

be a system of n functions, the system is seen as a function $f: \mathbb{R}^k \rightarrow \mathbb{R}^n$. The matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_k} \end{bmatrix}$$

is the **Jacobian** matrix. The matrix represents the derivative of f . When $n = 1$ then J is the gradient of the function f

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \right]$$

When $n = k$ then J is a squared matrix and has a determinant $\det(J)$. The determinant represents a relation between two variables $(y_1, \dots, y_n)$ and $(x_1, \dots, x_k)$.

- If $\det(J) \neq 0$ at a point $a \in \mathbb{R}^n$ then according to the inversion function theorem, f is invertible².
- If $\det(J) > 0$ at a point a then, f preserves the orientation.
- If $\det(J) < 0$ at a point a then, f does not preserve(inverses) the orientation.

The same idea is used in the implicit function theorem. Let $F(x, y) = 0$ be an implicit relation between x and y .

If the condition $\frac{\partial F}{\partial y} \neq 0$ is verified then there is a functional relation between x and y .

we can even find the derivative. Let $F(x, y) = 0$ then $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$ which gives $\frac{\partial F}{\partial x} dx = -\frac{\partial F}{\partial y} dy$ then

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

we have the derivative of the function $y = f(x)$.

The $\det(J)$ is also used in variables change in multiple integrals. Let's see the following example

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \end{cases}$$

Then $dx = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$ and $dy = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv$ which gives

$$dxdy = \left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv \right) \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv \right)$$

² The invariance of domain theorem due to Dutch mathematician Luitzen **Brouwer** (1881 – 1966) in 1912, says that if $n \neq m$ then there is neither a diffeomorphism nor a homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$. Although there may exist a function f and its inverse f^{-1} between the two spaces.

$$= \frac{\partial f}{\partial u} \frac{\partial g}{\partial u} du du + \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} du dv + \frac{\partial f}{\partial v} \frac{\partial g}{\partial u} dv du + \frac{\partial f}{\partial v} \frac{\partial g}{\partial v} dv dv$$

$$= \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} dudv - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u} dudv$$

$$= \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} dudv$$

which gives $dx dy = \det(J) du dv$. The number $\det(J)$, the determinant of the Jacobian matrix, is the scaling factor, thus we must take the positive $|\det(J)|$. If the determinant $\det(J) = 0$, then there is a dependence between the functions $(y_1, \dots, y_n)$. This also applies to linear systems, let

$$\begin{aligned} y_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + \cdots + a_{2n}x_n \\ &\vdots \\ y_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

be a linear system , the calculation of the **Jacobian** matrix gives

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

We know from linear algebra that if $\det(J) = 0$ the rows are linearly dependent.

○ First method

The first method rests on **Leibnitz** rule of derivation under integration. Let the integral $F(t) = \int f(x, t) dx$ then

$$F'(t) = \int \frac{\partial(x, t)}{\partial t} dx$$

So we take as our function $F(t)$ the integral

$$F(t) = \int_0^{+\infty} \frac{e^{-t^2(1+x^2)}}{1+x^2} dx$$

by derivation under the integral sign we find

$$F'(t) = -2te^{-t^2} \int_0^{+\infty} e^{-t^2x^2} dx$$

We change : $u = tx$ and $du = tdx$ we find

$$F'(t) = -2e^{-t^2} \int_0^{+\infty} e^{-u^2} du$$

By integration from 0 to $+\infty$, we have

$$\int_0^{+\infty} F'(t) dt = -2 \int_0^{+\infty} e^{-t^2} \left(\int_0^{+\infty} e^{-u^2} du \right) dt = \left(\int_0^{+\infty} e^{-u^2} du \right)^2$$

For the left side we easily get

$$\int_0^{+\infty} F'(t) dt = F(t)|_0^{+\infty} = -\frac{\pi}{2}$$

as $F(0) = \int_0^{+\infty} \frac{dt}{1+t^2}$ and $F(+\infty) = 0$. The final result then is

$$\left(\int_0^{+\infty} e^{-u^2} du \right)^2 = \frac{\pi}{4}$$

the result follows.

○ Second method

We know that the double integral implying the function $z = e^{-x^2-y^2}$ is a volume, then it is seen as an infinity of infinitesimal solid cylinders (or disks of height dz , and radius R_z). So to find the volume

$$V = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy$$

We sum the volumes of all cylinders, where the volume of each disk is

$$dV = \pi(R_z)^2 dz$$



A cylinder of radius R_z

The R_z means that the radii R of the cylinders depend on z . The infinite sum $\sum \pi(R_z)^2 dz$ which is the volume V , becomes a simple integration from $z = 0$ to $z = 1$

$$V = \int_0^1 \pi(R_z)^2 dz = \pi \int_0^1 (R_z)^2 dz$$

But we first find the expression of R_z . We see that z is simply : $z = e^{-(R_z)^2}$, or

$$R_z = \sqrt{-\log(z)}$$

which gives

$$V = -\pi \int_0^1 \log(z) dz$$

The anti-derivative of $\log(x)$ is $x\log(x) - x$ then

$$x\log(x) - x \Big|_0^1 = -1$$

As $\lim_{x \rightarrow 0} (x\log(x)) = 0$, which gives

$$V = \pi$$

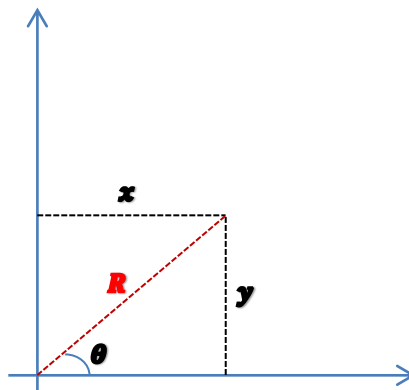
then $I = \sqrt{V} = \sqrt{\pi}$.

○ Third method

We notice immediately that we can take advantage from the presence of the exponent x^2 in the function, by linking it to the radius R of a circle, thus introducing polar coordinates to simplify the calculation. The change consists in the polar transformation : $(x, y) \rightarrow (R, \theta)$. We put

$$R^2 = x^2 + y^2$$

the equation of a circle of radius R



the relation between (x, y) and (R, θ) is

$$\begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta) \end{cases}$$

where R changes from 0 to $+\infty$ and the angle θ changes from 0 to 2π . So the problem is how to express $dx dy$ in terms of $dR d\theta$



The area ds of rectangle in (x, y) is obviously $ds = dx dy$. For the area of the corresponding shape in polar form (R, θ) , we have the width dR and the arc length $R d\theta$ so the area is $R dR d\theta$. Because the length of a circle arc of radius R and angle α is $R\alpha$ where α is in radian. Once this done we can now write that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$$

Becomes

$$\int_0^{2\pi} \int_0^{+\infty} e^{-R^2} R dR d\theta = I^2$$

This change of variables can be accomplished by using the Jacobian matrix J and its

determinant $\det(J)$. We return to our integral and in our example we have

$$\begin{cases} x = R\cos(\theta) \\ y = R\sin(\theta) \end{cases}$$

then

$$J = \begin{bmatrix} \cos(\theta) & -R\sin(\theta) \\ \sin(\theta) & R\cos(\theta) \end{bmatrix}$$

and its determinant is $|\det(J)| = R$. So the differential $dxdy$ in the polar form (R, θ) is $|\det(J)|dRd\theta$, and the integral is as before

$$\int_0^{2\pi} \left(\int_0^{+\infty} e^{-R^2} R dR \right) d\theta = \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-R^2} R dR$$

The inside integral gives

$$\int_0^{+\infty} e^{-R^2} R dR = \frac{1}{2} e^{-R^2} \Big|_0^{+\infty} = \frac{1}{2}$$

The second is

$$\int_0^{2\pi} d\theta = 2\pi$$

which gives the result π . So $I^2 = \pi$ and this gives $I = \sqrt{\pi}$.

Fourth method

This fourth method depends on the change $y = tx$ with $dy = xdt$ then in $\int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dxdy$ we find

$$\int_0^{+\infty} \int_0^{+\infty} e^{-x^2(1+t^2)} x dt dx$$

In order to get rid of the difficult function $e^{-x^2(1+t^2)}$ by integration, we multiply now by the factor $-2(1+t^2)$ and we find

$$\int_0^{+\infty} \frac{1}{-2(1+t^2)} dt \int_0^{+\infty} e^{-x^2(1+t^2)} (-2x(1+t^2)) dx$$

which gives

$$\frac{1}{-2} \int_0^{+\infty} \frac{1}{(1+t^2)} dt \left(e^{-x^2(1+t^2)} \Big|_0^{+\infty} \right) = \frac{1}{-2} \int_0^{+\infty} \frac{1}{1+t^2} dt = \frac{\pi}{4}$$

But we know that $I^2 = 4 \frac{\pi}{4}$ which gives then $I = \sqrt{\pi}$

Fifth method

This third method consists in using the Gamma function Γ . The Gamma function is by definition the following improper integral

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$$

The idea behind the above definition is to find a function F of real variable x such that

$$F(x = n) = n!$$

The solution F was first found by **Euler** (1707 – 1783) in 1729. We know the value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, which comes from the relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

so when plug the value $x = \frac{1}{2}$, we find the result

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} e^{-t} t^{-1/2} dt = \sqrt{\pi}$$

So to calculate I we perform a change of variable in our integral $\int_0^{+\infty} e^{-x^2} dx$. We put $x = \sqrt{t}$ then $dx = \frac{1}{2\sqrt{t}} dt$, from which we get

$$\int_0^{+\infty} e^{-x^2} dx = \frac{1}{2} \int_0^{+\infty} e^{-t} t^{-1/2} dt = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

and at last $I = 2 \int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.

Complex triangular inequality

Talking on complex numbers e^{ix} , I take the occasion to introduce here a method by which we can prove the triangular inequality for complex numbers $z = x + iy$.

First

$$|cz| = \sqrt{(cx)^2 + (cy)^2} = |c||z|$$

so this gives

$$|wz| = |w||z|e^{i(\theta_1 + \theta_2)} = |w||z|$$

So we have

$$\begin{aligned} |z + w| &= |z| \left| 1 + \frac{w}{z} \right| = |z| \left| 1 + \frac{|w|}{|z|} e^{i(\theta_1 - \theta_2)} \right| \\ &= |z| \left| 1 + \frac{|w|}{|z|} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \right| \\ &= |z| \sqrt{\left(1 + \frac{|w|}{|z|} \cos(\theta_1 - \theta_2) \right)^2 + \left(\frac{|w|}{|z|} \sin(\theta_1 - \theta_2) \right)^2} \\ &= |z| \sqrt{1 + 2 \frac{|w|}{|z|} \cos(\theta_1 - \theta_2) + \frac{|w|^2}{|z|^2}} \end{aligned}$$

$$\leq |z| \sqrt{\left(1 + \frac{|w|}{|z|}\right)^2}$$

$$= |z| \left|1 + \frac{|w|}{|z|}\right| = ||z| + |w|| = |z| + |w|$$

Exercices

1) Show that

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

1) Is the integral $\int x^2 e^{-x^2} dx$ elementary?

1) show that the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} (-1)^n$ converges.

1) Show the following :

* $Error(0) = 0$

* $Error(\infty) = 1$

* $Error(-t) = -Error(t)$

1) using implicit function theorem tell if there is a relation between y and x :

* $x^2 + y^2 = 2$

* $y^2 + x = 1$

* $x^2 + y^2 = x$

* $y^2 + x = y$

1) find $\frac{dy}{dx}$ in the following cases:

* $x^2 + y^3 = 1$

* $y^2 + x = 1$

* $x^3 + y^2 = x$

* $y^2 + x = y^3$

1) Using Leibnitz derivation method evaluate the following improper integral :

$$F(t) = \int_0^{+\infty} e^{-x} \frac{\sin(tx)}{x} dx$$

1) say how to calculate the limit

$$\lim_{x \rightarrow 0} (x \log(x) - x)$$

1) Evaluate the following double integrals :

$$* \int_0^2 \int_{-1}^4 xy dx dy$$

$$* \int_0^1 \int_0^3 (x^2 + y^2) dx dy$$

$$* \int_1^2 \int_1^2 \frac{dx dy}{x^2 + y^2}$$

1) Evaluate the difference between the two numbers : $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$ and $\int_{-\infty}^{+\infty} e^{-x^2} dx$

1) Evaluate the following integrals :

$$* \int_{-1}^1 \frac{dt}{t^2}$$

$$* \int_{-1}^1 \frac{dt}{t^3}$$

$$* \int_0^{+\infty} t \sin(t) dt$$

$$* \int_0^1 \frac{dt}{\sqrt[3]{t}}$$

$$* \int_1^5 \frac{\log(x)}{x} dx.$$

$$* \int_0^{+\infty} e^{-\sqrt{x}} dx.$$

1) let $a, b > 0$, Calculate :

$$\int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x} dx$$

1) Find the following anti-derivatives :

$$* \int \sqrt{\frac{3+2x}{x-2}} dx$$

$$* \int \cos^2(x) dx$$

$$* \int \sin^2(x) dx$$

$$* \int \cos^3(x) dx$$

$$* \int \sin^3(x) dx$$

$$* \int x e^x dx$$

$$* \int x^2 e^x dx$$

$$* \int \frac{x^5}{x^3 - 1} dx$$

$$* \int \sqrt{2x - x^2} dx$$

$$* \int \frac{\cos(x)}{\sin^2(x)} dx$$

$$* \int \frac{\cos(x)}{1 + \cos(x)} dx$$

$$* \int \frac{\sin(x)}{1 + \sin(x)} dx$$

$$* \int \cos^2(x) dx$$

$$* \int \frac{1}{1+x^4} dx$$

$$* \int \arctan(x) dx$$

$$* \int x \arctan(x) dx$$

$$* \int \log(1+x^2) dx$$

$$* \int \frac{1}{1+x^3} dx$$

$$* \int \frac{\sqrt{4x+x^2}}{x^2} dx$$

$$* \int \frac{dx}{1+\cos(x)}$$

$$* \int \frac{dx}{1+\sin(x)}$$

$$* \int \frac{dx}{\cos^4(x)}$$

$$* \int \frac{\cos^3(x) dx}{\sin^4(x)}$$

$$* \int \frac{1}{x-\sqrt{x^2-1}} dx$$

$$* \int x \cos^2(x) dx$$

$$* \int \frac{2x-1}{\sqrt{x^2-x+1}} dx$$

$$* \int \frac{1}{\sqrt{x^2-x+1}} dx$$

$$* \int \frac{3x+1}{x^2-x+2} dx$$

$$* \int \frac{x^2 dx}{5-x^6}$$

$$* \int \frac{e^{2x} dx}{1+e^{2x}}$$

$$* \int \frac{dx}{\sqrt{1-3x^2}}$$

1) Show that :

$$||z| + |w|| = |z| + |w|$$

1) Show that :

$$\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$$

1) Show that :

$$|z_1 + \dots + z_n| \leq |z_1| + \dots + |z_n|$$

1) Show that :

$$|z - w| \leq |z| + |w|$$

1) Show that :

$$|z_1 - \dots - z_n| \leq |z_1| + \dots + |z_n|$$

1) Show that :

$$|z| - |w| \leq |z - w|$$

1) Show that :

$$|z| - |w_n| \leq |z - w_1| + |w_1 - w_2| + \dots + |w_{n-1} - w_n|$$

Any mistake made here is mine